Uniform Amenability at Infinity

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2026 May

$\Gamma \curvearrowright \Gamma$ by left translation $\rightsquigarrow (sf)(x) = f(s^{-1}x)$ for a function f on Γ.
$\operatorname{Prob}\Gamma \subset \ell_1\Gamma$, equipped with the norm topology (= pointwise conv topo).
$\lambda\colon \Gamma \curvearrowright \ell_2\Gamma$ left reg repn, $\lambda(s)f = sf \rightsquigarrow \lambda\colon \mathbb{C}\Gamma \to \mathbb{B}(\ell_2\Gamma)$, $\lambda(f)g = f * g$.
$\mathrm{C}^*_\lambda\Gamma :=$ norm-closure $\lambda(\mathbb{C}\Gamma) \subset \mathbb{B}(\ell_2\Gamma)$, the reduced group C*-algebra.

Theorem (von Neumann, Reiter, Hulanicki, Følner, Kesten, . . .)

Γ is *amenable* if the following equivalent conditions hold.

- $\exists$ Γ-invariant finitely additive prob measure on Γ (*invariant mean*); i.e., $\exists\, m\colon \ell_\infty\Gamma \to \mathbb{C}$ positive unital linear functional that is Γ-invariant.
- $\exists\, \mu_i \in \operatorname{Prob}\Gamma$ such that $\forall s\ \lim_i \|\mu_i - s\mu_i\|_1 = 0$ (*approx inv mean*); i.e., $\forall F \Subset \Gamma\, \forall \varepsilon > 0\, \exists \mu \in \operatorname{Prob}\Gamma.\ \sum_{s\in F} \|\mu - s\mu\|_1 < \varepsilon$.
- $\|\lambda(\mu)\| = \lim_n \sqrt[2n]{\mu^{*2n}(1)} = 1$ for $\forall \mu \in \operatorname{Prob}\Gamma$, symmetric.

The class of amenable groups is stable under:
subgroups, quotients, extensions, directed unions.
$\rightsquigarrow$ finite grps, (v.) solvable grps, $\mathrm{Sym}(\infty) := \bigcup_n \mathrm{Sym}(n)$, $\mathrm{Alt}(5) \wr \mathbb{Z}$.
Non-cyclic free groups $\mathbf{F}$ are not amenable (Banach–Tarski Paradox).

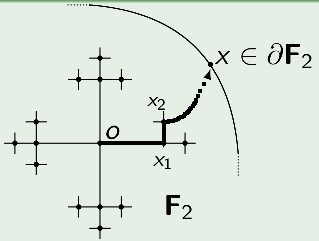
Many interesting groups are not amenable!

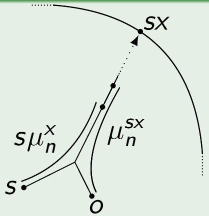
Definition (Anantharaman-Delaroche, AD–Renault (after Zimmer))

An action $\Gamma \curvearrowright X$ on a cpt topo space X is *topologically amenable* if $\exists \mu_i \colon X \to \operatorname{Prob}\Gamma$ **continuous** such that $\forall s\ \lim_i \sup_{x\in\Gamma} \|\mu_i^{sx} - s\mu_i^x\|_1 = 0$.

- $\mu\colon X \ni x \mapsto \mu^x \in \operatorname{Prob}\Gamma$ is continuous iff $\forall t\ x \mapsto \mu^x(t)$ is continuous. Moreover, w.m.a. by perturbation $\exists E \Subset \Gamma$ s.t. $\operatorname{supp}\mu^x \subset E$ for $\forall x$.
- If Γ is amenable, then every $\Gamma \curvearrowright X$ is topo amenable.
- If $\Gamma \curvearrowright X$ topo amen and $\operatorname{Prob}_\Gamma X \neq \emptyset$ (e.g., $X = \{\mathrm{pt}\}$), then Γ amen.

Prototype (Topo amenability of $\mathbf{F}_r \curvearrowright \partial\mathbf{F}_r \subset \{g_1, g_1^{-1} \ldots, g_r, g_r^{-1}\}^{\mathbb{N}}$)





For $x \in \partial\mathbf{F}_r$, put $\mu_n^x :=$ unif prob on the first n-segment of $[o,x]$. Then
$$\|\mu_n^{sx} - s\mu_n^x\|_1 \leq \frac{2|s|}{n}.$$
□

$\Gamma \curvearrowright X$ t.a. $\Leftrightarrow_{\text{def}} \exists \mu_i \colon X \to \operatorname{Prob}\Gamma$ cts s.t. $\forall s\ \lim_i \sup_x \|\mu_i^{sx} - s\mu_i^x\|_1 = 0$.

- If $\Gamma \curvearrowright X$ is topo amen and $Y \twoheadrightarrow_\Gamma X$ (i.e. $C(X) \hookrightarrow C(Y)$), then $\Gamma \curvearrowright Y$.
- If $\exists\, \Gamma \curvearrowright X$ topo amen, then $\Gamma \curvearrowright \beta\Gamma$ topo amen ($\Leftrightarrow \Gamma \curvearrowright \ell_\infty\Gamma$ amen).

Theorem (Anantharaman-Delaroche, Kirchberg–Wassermann, Ozawa, Guentner–Kaminker, Dykema, Adams, Guentner–Higson–Weinberger)

Γ is *exact*, a.k.a. *amenable at infinity*, if the following equiv cond hold.

- $\exists\, \Gamma \curvearrowright X$ topo amen, or equiv $\Gamma \curvearrowright \ell_\infty\Gamma$ is amenable.
- $\forall F \Subset \Gamma\, \forall \varepsilon > 0\, \exists E \Subset \Gamma\, \exists \mu \colon \Gamma \to \operatorname{Prob}\Gamma$ such that $\operatorname{supp}\mu^x \subset E$ and
$$\sum_{s \in F} \sup_{x \in \Gamma} \|\mu^{sx} - s\mu^x\|_1 < \varepsilon.$$
- $\mathrm{C}^*_\lambda\Gamma$ is *exact*, that is, $(\,\cdot\,) \otimes \mathrm{C}^*_\lambda\Gamma$ is exact.

The class of exact groups contains hyperbolic groups, linear groups, etc.

The class of exact groups is stable under:
subgroups, ~~quotients~~, extensions, directed unions,
amalgamated free products, HNN extensions.

Exactness is an important notion in Analytic Group Theory that has applications to the Baum–Connes conjecture (Yu, Tu, Higson, Kasparov, Roe,...), classification theory for group vN algebras, etc.

How can we estimate $\|\lambda(f)\|$?

Spectral radius formula

$$\|\lambda(f)\| = \lim_n \sqrt[2n]{(f^* * f)^{*2n}(1)}$$

requires global info and is not "computable" in the algorithmic sense.

For $E \Subset \Gamma$, let $\Phi_E\colon \mathbb{B}(\ell_2\Gamma) \to \mathbb{B}(\ell_2 E)$ denote the compression, given by

$$\Phi_E(T) := P_{\ell_2 E} T|_{\ell_2 E}.$$

A group Γ is exact iff $\forall F \Subset \Gamma\, \forall \varepsilon > 0\, \exists E \Subset \Gamma\, \exists \mu\colon \Gamma \to \operatorname{Prob} E$ such that

$$\sum_{s\in F} \sup_{x\in\Gamma} \|\mu^{sx} - s\mu^x\|_1 < \varepsilon.$$

$$\rightsquigarrow \quad (1-\varepsilon)\|\lambda(f)\| \le \|\Phi_E(\lambda(f))\|_{\mathbb{B}(\ell_2 E)} \le \|\lambda(f)\|$$

for every $f \in \mathrm{C}^*_\lambda\Gamma$ with $\operatorname{supp} f \subset F$ (and $f \in A \otimes \mathrm{C}^*_\lambda\Gamma$ with A arbitrary).

$\because$ $V\colon \ell_2\Gamma \to \ell_2 E \otimes \ell_2\Gamma$, defined by $V\delta_x := \sum_t \mu^x(xt^{-1})^{1/2}\delta_{xt^{-1}} \otimes \delta_t$, is an isometry satisfying $V^*(\Phi_E(\lambda(s)) \otimes 1)V = \lambda(s)D_s$ in $\mathbb{B}(\ell_2\Gamma)$ with $D_s := \operatorname{diag}_x\langle s(\mu^x)^{1/2}, (\mu^{sx})^{1/2}\rangle \approx 1$ for $s \in F$. $\square$

$\rightsquigarrow$ One can recover $\lambda(F)$ on $\ell_2\Gamma$ from the partially def'd action $F \curvearrowright E$ and $\|\lambda(f)\|$ is "computable" modulo μ.

$\mathcal{U} \subset \mathfrak{P}(\mathbb{N})$ a free ultrafilter on $\mathbb{N}$ (all arguments indep of the choice of $\mathcal{U}$)

- $\mathbb{N} \in \mathcal{U}$ • $A \in \mathcal{U}$, $A \subset B \Rightarrow B \in \mathcal{U}$ • $\forall A.\ A \in \mathcal{U}$ or $A^{c} \in \mathcal{U}$
- $\mathcal{U}$ is *free* if $\nexists n$ such that $(A \in \mathcal{U} \Leftrightarrow n \in A)$.

$\rightsquigarrow$ a hom $\mathrm{Lim}_{\mathcal{U}} \colon \ell_\infty\mathbb{N} \to \mathbb{C}$; $\quad a = \mathrm{Lim}_{\mathcal{U}}\, a_n$ iff $\{n : |a_n - a| < \varepsilon\} \in \mathcal{U}$.

$X^{\mathcal{U}} := \prod_{\mathbb{N}} X/\!\sim$, where $(x_n)_n \sim (y_n)_n$ if $\{n : x_n = y_n\} \in \mathcal{U}$.
Write $[x_n]_n \in X^{\mathcal{U}}$ for the equiv class for $(x_n)_n$.

- $\exists E_n \Subset X$ w. $|E_n| = k$ and $E = \{[x_n]_n : x_n \in E_n\} \Leftrightarrow E \Subset X^{\mathcal{U}}$ w. $|E| = k$.

Theorem (Keller, Bożejko, Wysoczański, ...)

Γ is *uniformly amenable* if the following equivalent conditions hold.

- $\Gamma^{\mathcal{U}}$ is amenable.
- $\exists k \colon \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ (*modulus of u.a.*) $\forall F \Subset \Gamma\ \forall \varepsilon > 0\ \exists \mu \in \mathrm{Prob}\,\Gamma$
  $|\operatorname{supp}\mu| \le k(|F|, \varepsilon)$ and $\sum_{s \in F} \|\mu - s\mu\|_1 < \varepsilon$.
- $\lim_n \sqrt[2n]{\nu^{*2n}(1)} = 1$ *uniformly* for $\nu \in \mathrm{Prob}\,\Gamma$ symmetric, $|\operatorname{supp}\nu|$ bdd.

The class of uniformly amenable groups is stable under:
subgroups, quotients, extensions, ~~directed unions~~.
$\rightsquigarrow$ finite grps, (v.) solvable grps, $\cancel{\mathrm{Sym}(\infty) := \bigcup_n \mathrm{Sym}(n)}$, $\mathrm{Alt}(5) \wr \mathbb{Z}$.

OPEN: ¿ $\Gamma^{\mathcal{U}}$ free-subgroup-free $\Rightarrow$ Γ unif amen ?

When $\Gamma^{\mathcal{U}}$ exact?

Γ hyperbolic (Kharlampovich–Myasnikov & Sela)

or linear, i.e., subgroups of $\mathrm{GL}_d(K)$ (Guentner–Higson–Weinberger).

This fact alone does **not** seem useful. We need a stronger property.

Theorem (by the traditional recipe)

Γ is *uniformly exact*, or *u.a. at infinity*, if the following equiv cond hold.

- $\Gamma^{\mathcal{U}} \curvearrowright (\ell_\infty\Gamma)^{\mathcal{U}}$ amenable (as opposed to $\Gamma^{\mathcal{U}} \curvearrowright \ell_\infty(\Gamma^{\mathcal{U}})$).
- $\exists k\colon \mathbb{N}\times\mathbb{R}_+ \to \mathbb{N}$ (*modulus of u.e.*) $\forall F \Subset \Gamma\, \forall\varepsilon>0\, \exists\mu\colon \Gamma \to \mathrm{Prob}\,\Gamma$ $|\bigcup_{x\in\Gamma} \mathrm{supp}\,\mu^x| < k(|F|,\varepsilon)$ and $\sum_{s\in F} \sup_{x\in\Gamma} \|\mu^{sx} - s\mu^x\|_1 < \varepsilon$.
- The same as above, but $\bigcup_{x\in\Gamma} \mathrm{supp}\,\mu^x \subset (F\cup\{1\}\cup F^{-1})^{k(|F|,\varepsilon)}$.
- $A^{\mathcal{U}} \otimes \mathrm{C}^*_\lambda(\Gamma^{\mathcal{U}}) \hookrightarrow (A\otimes \mathrm{C}^*_\lambda\Gamma)^{\mathcal{U}}$ for every C*-algebra A.

For a C*-algebra A, the ultrapower is

$$A^{\mathcal{U}} := \ell_\infty(\mathbb{N},A)/\{(a_n)_n : \mathrm{Lim}_{\mathcal{U}}\|a_n\| = 0\}.$$

- $(\ell_\infty\Gamma)^{\mathcal{U}} \subset \ell_\infty(\Gamma^{\mathcal{U}})$, $[f_n]_n([x_n]_n) := \mathrm{Lim}_{\mathcal{U}} f_n(x_n)$, as “internal” functions, but $(\mathrm{C}^*_\lambda\Gamma)^{\mathcal{U}}$ on $(\ell_2\Gamma)^{\mathcal{U}}$ and $\mathrm{C}^*_\lambda(\Gamma^{\mathcal{U}})$ on $\ell_2(\Gamma^{\mathcal{U}})$ are *a priori* unrelated.

Conjecture: Hyperbolic groups and linear groups are uniformly exact.

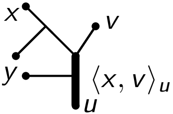
Theorem (O. 2026)

Free groups $\mathbf{F}$ are uniformly exact.

- $\exists k\colon \mathbb{N}\times\mathbb{R}_+\to\mathbb{N}$ (*modulus of u.e.*) $\forall f\in \mathrm{C}^*_\lambda\Gamma$ with $\operatorname{supp} f\subset F\Subset\Gamma$
$$(1-\varepsilon)\|\lambda(f)\|\leq\|\Phi_E(\lambda(f))\|_{\mathbb{B}(\ell_2E)}\leq\|\lambda(f)\|$$
for $E=(F\cup\{1\}\cup F^{-1})^{k(|F|,\varepsilon)}$. Unif conv for the spectral radius formula
$$\|\lambda(f)\|=\lim_n\sqrt[2n]{(f^* * f)^{*2n}(1)}.$$
- An amenable group is uniformly exact iff it is uniformly amenable.
$\rightsquigarrow$ $\mathrm{Sym}(\infty)$ is not uniformly exact.
- The class of u.e. grps is stable under subgrps, extensions, free products.
$\because\quad (\Gamma/\Delta)^{\mathcal{U}}\cong\Gamma^{\mathcal{U}}/\Delta^{\mathcal{U}};\quad \mathbf{1}\to\mathbf{F}\to\Gamma*\mathbb{Z}\to\Gamma\to\mathbf{1}.$
- The class of u.e. grps having the same m.o.u.e. is closed in the space of marked grps. Convergence of a seq of marked grps having the same m.o.u.e. implies *strong* convergence in the operator algebraic sense.
$\rightsquigarrow$ Limit grps are the strong limit of free grps (Louder–Magee 2022).

OPEN: ¿ Amalgamated free products and HNN extensions ?

- A sketch of the proof of u.e. for a f.g. free group $\mathbf{F}$.



$\mathbf{F}$ with the standard metric is a *tree* and $\mathbf{F}^{\mathcal{U}}$ is a $\mathbb{Z}^{\mathcal{U}}$-*tree*; $\mathrm{d}_{\mathcal{U}}(x,y) := [\mathrm{d}(x_n,y_n)]_n \in \mathbb{Z}^{\mathcal{U}}$ for $x=[x_n]_n$ and $y=[y_n]_n$.

$\langle x, v\rangle_u := \frac{1}{2}(\mathrm{d}(x,u)+\mathrm{d}(v,u)-\mathrm{d}(x,v))$. $\qquad \langle x, v\rangle_u \geq \langle x, y\rangle_u \wedge \langle y, v\rangle_u$

$\Lambda := \mathbb{Z}^{\mathcal{U}}$ is an ordered abelian group.
For $a, b \in \Lambda_+$, write $b \preceq a$ if $\exists N\ b < Na$; and $b \ll a$ if $b \preceq a$ and $a \not\preceq b$.
For $a \in \Lambda_+$, $\exists!\ \varepsilon^a \colon \{b : |b| \preceq a\} \to \mathbb{R}$, $a \mapsto 1$. One has $\varepsilon^a(b) = \mathrm{Lim}_{\mathcal{U}} \frac{b_n}{a_n}$.

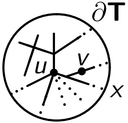
- For amenability of $\mathbf{F}^{\mathcal{U}} \curvearrowright (\ell_\infty \mathbf{F})^{\mathcal{U}}$, we may pass to a f.g. $\Gamma = \langle F\rangle \leq \mathbf{F}^{\mathcal{U}}$.

Fix $z=[z_n]_n \in \mathbf{F}^{\mathcal{U}}$ and set $c := \sum_{s\in F} \mathrm{d}_{\mathcal{U}}(z,sz) = [\sum_{s\in F} \mathrm{d}(z_n,s_nz_n)]_n \in \Lambda$.

$$\mathbf{T} := \{x \in \mathbf{F}^{\mathcal{U}} : \mathrm{d}(x,z) \preceq c\}/\!\sim_{\mathrm{d}} \text{ with } \mathrm{d}(x,y) := \varepsilon^c(\mathrm{d}_{\mathcal{U}}(x,y)) \in \mathbb{R}$$

is an $\mathbb{R}$-*tree* on which Γ acts isometrically.

$h_{u,v}(x) := \frac{1}{\mathrm{d}(u,v)}\langle x, v\rangle_u$ and $\mathcal{C}(\mathbf{T}) := \mathrm{C}^*(\{h_{u,v} : u, v \in \mathbf{T}\})$.



The Gelfand spectrum $\hat{\mathbf{T}}$ of $\mathcal{C}(\mathbf{T})$ is the *tree compactification*.
$\rightsquigarrow$ $\hat{\mathbf{T}} = \mathbf{T} \cup \partial\mathbf{T}$, for the visual boundary $\partial\mathbf{T}$, and with topo gen'd by $\{x \in \hat{T} : \langle x, v\rangle_u > 0\} = \{x \in \hat{T} : u \notin [v,x]\}$.

$h_{u,v}(x) = \varepsilon^{\mathrm{d}(u,v)}(\langle x, v\rangle_u)$ is def'd on $\mathbf{F}^{\mathcal{U}}$ and $h_{u,v} = [h_{u_n,v_n}]_n \in (\ell_\infty \mathbf{F})^{\mathcal{U}}$.

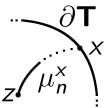
Key obs: $\Gamma = \langle F \rangle \curvearrowright \hat{\mathbf{T}}$ is topo amen modulo stabilizers at branch points.

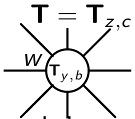
Theorem (Anantharaman-Delaroche, AD–Renault)

An action $\Gamma \curvearrowright X$ of a **c'ble** grp Γ on a **cpt** space X is topo amen iff $\exists \mu_i \colon X \to \operatorname{Prob}\Gamma$ **Borel**. $\forall s \forall m \in \operatorname{Prob} X$ $\lim_i \int \|\mu_i^{sx} - s\mu_i^x\|_1 \, dm(x) = 0$.

We may pass to a subtree and assume that $\mathbf{T}$ is a separable $\mathbb{R}$-tree.

- The countable Borel equiv rel $\mathcal{R}_{\Gamma \curvearrowright \mathbf{T}}$ is $\mathcal{M}$-amenable.
  $\because$ $\mathcal{R}_{\Gamma \curvearrowright \mathbf{T}}$ is a countable directed union of interval exchange trans o.e.'s.
- Except for branch points, stabilizer subgrps are amenable.
  $\because$ Arc stabilizers are abelian for limit $\mathbb{R}$-trees.
- $\{$branch points$\}$ is countable and $\{$amenable subgrps of $\Gamma\}$ is countable.
  (Kharlampovich–Myasnikov & Sela)
  $\rightsquigarrow \Gamma \curvearrowright \hat{T} \setminus \{$branch points$\}$ is Borel amenable.





- Amenability of $\Gamma \curvearrowright (\ell_\infty \mathbf{F})^{\mathcal{U}}$ reduces to that for

$\Gamma_w = \{t \in \Gamma : tw = w \text{ in } \mathbf{T}\} \curvearrowright \{y \in \mathbf{F}^{\mathcal{U}} : \mathrm{d}_{\mathcal{U}}(y, w) \ll c\}^{\wedge}$ for each b.p. w.

- We can make this process terminate as $\mathbf{F}$ has *tame geom at infinity*. $\square$

For (G, ℓ) and $s, t \in G^{\mathcal{U}}$, set $s \approx t$ if $\exists N$ $\frac{1}{N}\ell^{\mathcal{U}}(s) < \ell^{\mathcal{U}}(t) < N\ell^{\mathcal{U}}(s)$ in $\mathbb{Z}^{\mathcal{U}}$.
G has t.g.a.i. if $\forall d \, \exists r \, \forall d$-generated subgrp $\Gamma \leq G^{\mathcal{U}}$. $|\Gamma/{\approx}| \leq r$.

¿Estimate for the modulus of u.e.? ¿ Hyperbolic groups ?